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ORIGINAL PAPER

NTU-bankruptcy problems: consistency and the relative adjustment principle

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Abstract

This paper axiomatically studies bankruptcy problems with nontransferable utility by focusing on generalizations of consistency and the contested garment principle. On the one hand, we discuss several consistency notions and introduce the class of parametric bankruptcy rules which contains the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule. On the other hand, we introduce the class of adjusted bankruptcy rules and characterize the relative adjustment principle by truncation invariance, minimal rights first, and a weak form of relative symmetry.

Keywords NTU-bankruptcy problems · Consistency · Relative adjustment principle · Parametric bankruptcy rules · Adjusted bankruptcy rules

JEL Classification C79 · D63 · D74

1 Introduction

A bankruptcy problem with nontransferable utility, shortly an NTU-bankruptcy problem, arises when claimants have individual and incomparable claims on a set of attainable utility allocations. Bankruptcy rules assign to each such a bankruptcy problem a feasible utility allocation. NTU-bankruptcy problems form a natural gener-

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alization of TU-bankruptcy problems where the assumption of linear and transferable utility is dropped. TU-bankruptcy problems are well-studied in the literature (cf. Thomson 2003, 2013, 2015) and the question arises whether and how bankruptcy theory can be extended to NTU-bankruptcy problems. However, this passage is in general fraught with difficulties.

Orshan et al. (2003), and Dietzenbacher (2018), Estévez-Fernández et al. (2019) studied NTU-bankruptcy problems from a game theoretic perspective by defining an appropriate coalitional bankruptcy game and focusing on the structure of the core. Instead, we continue on the axiomatic approach of Dietzenbacher et al. (2016) by formulating some appropriate properties for bankruptcy rules and studying their implications.

Dietzenbacher et al. (2016) explored proportionality, equality, and duality in the context of NTU-bankruptcy problems and introduced the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule. They extended axiomatic characterizations by adequately generalizing the corresponding properties for TU-bankruptcy rules to NTU-bankruptcy rules. In particular, they defined the relative symmetry property which imposes a relatively equal treatment of claimants with relatively equal claims, i.e. equal claims in relation to their utopia values. Moreover, they defined the property of truncation invariance, which imposes invariance of the prescribed allocation under truncation of the claims by the utopia values.

For bankruptcy problems with transferable utility, the proportional rule, the constrained equal awards rule, and the constrained equal losses rule can be considered as the three basic bankruptcy rules. Herrero and Villar (2001) called these bankruptcy rules the three musketeers. Another well-studied rule for bankruptcy problems with transferable utility, which according to Herrero and Villar (2001) plays the role of D'Artagnan, is the so-called Talmud rule. Aumann and Maschler (1985) showed that the Talmud rule is the unique TU-bankruptcy rule satisfying consistency and the contested garment principle. This paper offers a first, careful attempt to generalize these two concepts to bankruptcy problems with nontransferable utility on which a generalized Talmud rule can be based in future research.

Following Thomson (2011), the consistency principle can be stated as follows. Consider a bankruptcy problem and the corresponding payoff allocation assigned by a particular bankruptcy rule. Suppose that some claimants leave with their allocated payoffs and that the remaining claimants reevaluate their allocated payoffs. The bankruptcy rule is called consistent if it prescribes for this reduced problem the same payoffs for the involved claimants. The design of these reduced problems for NTU-bankruptcy problems is however not straightforward, and different modeling choices have different consequences.

We examine the relation of the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule with several consistency notions. The proportional rule satisfies a multilateral consistency notion which converts reduced problems into new bankruptcy problems for the remaining claimants. This result can be used to derive new axiomatic characterizations using an elevator lemma. The constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy multilateral consistency, but they do satisfy consistency on a restricted domain which includes NTU-bankruptcy problems induced by TU-bankruptcy problems. Inspired by Young (1987), we introduce a class of parametric bankruptcy rules which contains the three basic bankruptcy rules, and we show that all parametric bankruptcy rules satisfy a consistency notion which interprets the reduced problem as the original bankruptcy problem where the leaving claimants leave a footprint behind. Future research could further explore this footprint consistency notion and the class of parametric bankruptcy rules.

The contested garment principle for TU-bankruptcy rules describes a standard solution for bankruptcy problems with two claimants where they first concede the minimal rights to each other and subsequently divide the remaining estate equally. To adequately generalize this two-claimant solution to the relative adjustment principle for NTU-bankruptcy rules, we study minimal rights in NTU-bankruptcy problems. The minimal rights first property requires that first allocating minimal rights, the maximal individual payoffs within the estate when all other claimants are allocated their claims, and subsequently applying the bankruptcy rule to the remaining bankruptcy problem, leads to the same payoff allocation as direct application of the bankruptcy rule to the original bankruptcy problem.

The three basic bankruptcy rules do not satisfy minimal rights first. Inspired by Thomson and Yeh (2008), we introduce the truncation operator and minimal rights operator which 'force' bankruptcy rules to satisfy truncation invariance and minimal rights first, respectively. The new bankruptcy rules obtained by applying both operators to existing ones form the class of adjusted bankruptcy rules. All adjusted counterparts of bankruptcy rules which satisfy relative symmetry coincide on the class of bankruptcy problems with two claimants. The corresponding two-claimant NTU-bankruptcy rule is called the relative adjustment principle which generalizes the contested garment principle for TU-bankruptcy problems. The new principle is characterized by truncation invariance, minimal rights first, and a restricted form of relative symmetry.

This paper is organized in the following way. In Sect. 2, we provide an overview of NTU-bankruptcy theory. Section 3 discusses several consistency notions and introduces the class of parametric bankruptcy rules. Section 4 introduces the class of adjusted bankruptcy rules and studies the relative adjustment principle.

2 Preliminaries

Let N be a nonempty and finite set of *claimants*. The collection of all subsets of N is denoted by $2^N = \{S \mid S \subseteq N\}$. For any $x, y \in \mathbb{R}^N$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. For any set of payoff allocations $E \subseteq \mathbb{R}^N_+$,

- the *comprehensive hull* is given by $\text{comp}(E) = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in E} : y \geq x\}$;
- the *weak upper contour set* is given by $\text{WUC}(E) = \{x \in \mathbb{R}^N_+ \mid \neg\exists_{y \in E} : y > x\}$;
- the *weak Pareto set* is given by $\text{WP}(E) = \{x \in E \mid \neg\exists_{y \in E} : y > x\}$;
- the *strong Pareto set* is given by $\text{SP}(E) = \{x \in E \mid \neg\exists_{y \in E, y \neq x} : y \geq x\}$.

Note that $\text{SP}(E) \subseteq \text{WP}(E) \subseteq \text{WUC}(E)$. A set of payoff allocations $E \subseteq \mathbb{R}^N_+$ is called *comprehensive* if $E = \text{comp}(E)$, and *nonleveled* if $\text{SP}(E) = \text{WP}(E)$.

A *bankruptcy problem with nontransferable utility* (cf. Orshan et al. 2003) is a triple (N, E, c) in which $E \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, comprehensive, and nonleveled *estate*, and $c \in \mathrm{WUC}(E)$ is a vector of *claims*.[1] Let BR^N denote the class of all bankruptcy problems with claimant set N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \mathrm{BR}^N$.

Let $(E, c) \in \mathrm{BR}^N$. The vector of *utopia values* $u^E \in \mathbb{R}^N_+$ is given by

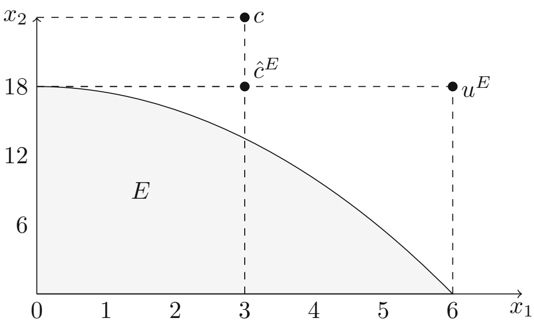
$$u^E = (\max\{x_i \mid x \in E\})_{i \in N}.$$

The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is given by

$$\hat{c}^E = \left(\min\{c_i, u_i^E\}\right)_{i \in N}.$$

Note that $(E, \hat{c}^E) \in \mathrm{BR}^N$.

Example 1 Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$. We have $u^E = (6, 18)$ and $\hat{c}^E = (3, 18)$. This is illustrated as follows.



△

A *bankruptcy rule* f on BR^N assigns to any $(E, c) \in \mathrm{BR}^N$ a payoff allocation $f(E, c) \in \mathrm{WP}(E)$ for which $f(E, c) \leq c$. A bankruptcy rule f on BR^N satisfies

- *relative symmetry* if $f_i(E, c)u_j^E = f_j(E, c)u_i^E$ for all $(E, c) \in \mathrm{BR}^N$ and any $i, j \in N$ with $c_i u_j^E = c_j u_i^E$;
- *truncation invariance* if $f(E, c) = f(E, \hat{c}^E)$ for all $(E, c) \in \mathrm{BR}^N$.

The *proportional rule* Prop on BR^N (cf. Dietzenbacher et al. 2016) assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{Prop}(E, c) = \lambda^{E,c} c,$$

where $\lambda^{E,c} \in [0, 1]$ is such that $\mathrm{Prop}(E, c) \in \mathrm{WP}(E)$. The proportional rule satisfies relative symmetry, but does not satisfy truncation invariance.

[1] Alternatively, one can interpret a bankruptcy problem with nontransferable utility as a bargaining problem with claims (cf. Chun and Thomson 1992) where the disagreement point equals the zero vector, or as a Nash rationing problem (cf. Mariotti and Villar 2005) where the admissible allocations are nonnegative. Contrary to these models, we allow for a nonconvex estate and claims which exceed the maximal individual payoffs within the estate.

The *constrained relative equal awards rule* CREA on BR^N (cf. Dietzenbacher et al. 2016) assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{CREA}(E, c) = \left(\min\{c_i, \alpha^{E,c} u_i^E\}\right)_{i \in N},$$

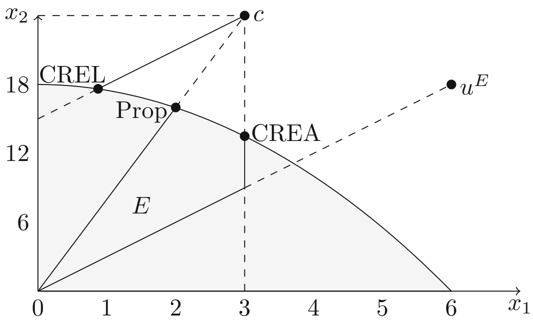
where $\alpha^{E,c} \in [0, 1]$ is such that $\mathrm{CREA}(E, c) \in \mathrm{WP}(E)$. The constrained relative equal awards rule satisfies both relative symmetry and truncation invariance.

The *constrained relative equal losses rule* CREL on BR^N (cf. Dietzenbacher et al. 2016) assigns to any $(E, c) \in \mathrm{BR}^N$ with $E \neq \{0_N\}$ the payoff allocation

$$\mathrm{CREL}(E, c) = \left(\max\{0, c_i - \beta^{E,c} u_i^E\}\right)_{i \in N},$$

where $\beta^{E,c} \in \mathbb{R}_+$ is such that $\mathrm{CREL}(E, c) \in \mathrm{WP}(E)$. The constrained relative equal losses rule satisfies relative symmetry, but does not satisfy truncation invariance.

Example 2 Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $\lambda^{E,c} = \frac{2}{3}$, $\alpha^{E,c} = \frac{3}{4}$, and $\beta^{E,c} = 1 - \frac{1}{6}\sqrt{15}$. This means that $\mathrm{Prop}(E, c) = (2, 16)$, $\mathrm{CREA}(E, c) = (3, 13\frac{1}{2})$, and $\mathrm{CREL}(E, c) = (\sqrt{15}-3, 3\sqrt{15}+6)$. This is illustrated as follows.



△

3 Consistency

Consistency requires that application of a bankruptcy rule to a reduced problem leads to the same payoffs for the involved claimants as within the original bankruptcy problem. For TU-bankruptcy problems, the estate of such a reduced problem can simply be defined as the original estate subtracted with the allocated payoffs to the leaving claimants (cf. Aumann and Maschler 1985). For NTU-bankruptcy problems, the design of such a reduced problem is not straightforward. We discuss several ways to generalize the consistency property for TU-bankruptcy rules.

A natural option is to convert the reduced problem into a new bankruptcy problem for the remaining claimants in which the estate is defined as the part of the original estate where all leaving claimants are allocated their corresponding payoffs. For this, we need to extend the domain of bankruptcy rules to bankruptcy problems for any nonempty subset of claimants.

Let $\overline{\mathrm{BR}}^N$ denote $\bigcup_{S\in 2^N\setminus\{\emptyset\}} \mathrm{BR}^S$. A bankruptcy rule f on $\overline{\mathrm{BR}}^N$ assigns to any $(E, c) \in \mathrm{BR}^S$ with $S \in 2^N\setminus\{\emptyset\}$ a payoff allocation $f(E, c) \in \mathrm{WP}(E)$ for which $f(E, c) \leq c$.

Let $(E, c) \in \mathrm{BR}^N$, let $x \in \mathbb{R}^N_+$, and let $S \in 2^N\setminus\{\emptyset\}$. The set of payoff allocations $E^x_S \subseteq \mathbb{R}^S_+$ is defined by

$$E^x_S = \left\{ y \in \mathbb{R}^S_+ \mid (y, x_{N\setminus S}) \in E \right\}.$$

Note that $(E_S^{f(E,c)}, c_S) \in \mathrm{BR}^S$ for any bankruptcy rule f on $\overline{\mathrm{BR}}^N$.

A bankruptcy rule is multilaterally consistent if it assigns to each reduced problem the same payoffs for the remaining claimants as within the original bankruptcy problem.

Definition 3.1 (*Multilateral consistency*) A bankruptcy rule f on $\overline{\mathrm{BR}}^N$ satisfies *multilateral consistency* if $f_S(E, c) = f(E_S^{f(E,c)}, c_S)$ for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N\setminus\{\emptyset\}$.

The weaker property which only considers reduced problems for two remaining claimants is called bilateral consistency.

Definition 3.2 (*Bilateral consistency*) A bankruptcy rule f on $\overline{\mathrm{BR}}^N$ satisfies *bilateral consistency* if $f_S(E, c) = f(E_S^{f(E,c)}, c_S)$ for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N$ with $|S| = 2$.

In other words, a bankruptcy rule is bilaterally consistent if it assigns to each two-claimant reduced problem the same payoffs for the remaining claimants as within the original bankruptcy problem. This principle can also be applied in reverse direction. Consider any bankruptcy problem and a corresponding feasible payoff allocation. Suppose that for each two-claimant reduced problem a bankruptcy rule prescribes the corresponding payoffs within this allocation. Then the rule is called conversely consistent (cf. Thomson 2011) if it assigns this payoff allocation to the original bankruptcy problem.

Definition 3.3 (*Converse consistency*) A bankruptcy rule f on $\overline{\mathrm{BR}}^N$ satisfies *converse consistency* if $f(E, c) = x$ for all $(E, c) \in \mathrm{BR}^N$ and any $x \in \mathrm{WP}(E)$ with $x \leq c$ for which $x_S = f(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$.

If a bilateral consistent rule coincides with a conversely consistent rule on the class of two-claimant bankruptcy problems, then the rules coincide for any bankruptcy problem. This type of result is known as an elevator lemma (cf. Thomson 2011).

Lemma 3.1 (Elevator Lemma) *Let f and g be two bankruptcy rules on $\overline{\mathrm{BR}}^N$. If f satisfies bilateral consistency, g satisfies converse consistency, and $f(E, c) = g(E, c)$ for all $(E, c) \in \mathrm{BR}^S$ with $S \in 2^N$ and $|S| = 2$, then $f = g$.*

Proof Let $(E, c) \in \mathrm{BR}^N$ and let $x = f(E, c)$. Since f satisfies bilateral consistency, we have $x_S = f(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$. This means that $x_S = g(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$. Since g satisfies converse consistency, this implies that $g(E, c) = x$. Hence, $f(E, c) = g(E, c)$. □

For a bankruptcy rule which satisfies both bilateral consistency and converse consistency, the Elevator Lemma can be used to extend axiomatic characterizations from bankruptcy problems with two claimants to problems with any number of claimants. An example of such a bankruptcy rule is the proportional rule.

Lemma 3.2 *The proportional rule satisfies multilateral consistency.*

Proof Let $(E, c) \in \mathrm{BR}^N$ and let $S \in 2^N \backslash \{\emptyset\}$. We have $\mathrm{Prop}_S(E, c) = \lambda^{E,c} c_S$ and

$$\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S) = \lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} c_S,$$

where $\lambda^{E,c} \in [0, 1]$ is such that $\mathrm{Prop}(E, c) \in \mathrm{WP}(E)$ and $\lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} \in [0, 1]$ is such that

$$\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{Prop}(E,c)}).$$

Since $\mathrm{Prop}_S(E, c) \in E_S^{\mathrm{Prop}(E,c)}$, we have $\mathrm{Prop}_S(E, c) \leq \mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S)$. Since E is nonleveled and

$$\left(\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S), \mathrm{Prop}_{N \backslash S}(E, c)\right) \in E,$$

this means that $\mathrm{Prop}_S(E, c) = \mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S)$. Hence, the proportional rule satisfies multilateral consistency. □

Lemma 3.3 *The proportional rule satisfies converse consistency.*

Proof Let $(E, c) \in \mathrm{BR}^N$ and let $x \in \mathrm{WP}(E)$ with $x \leq c$ be such that $x_S = \mathrm{Prop}(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$. We have $\mathrm{Prop}(E, c) = \lambda^{E,c} c$. Moreover, we have $x_S = \lambda^{E^x_S, c_S} c_S$ for all $S \in 2^N$ with $|S| = 2$, which means that $x = tc$ for some $t \in [0, 1]$. Since E is nonleveled, this means that $\mathrm{Prop}(E, c) = x$. Hence, the proportional rule satisfies converse consistency. □

Theorem 3.4 *Any axiomatic characterization of the proportional rule for two-claimant bankruptcy problems yields an axiomatic characterization of the proportional rule for bankruptcy problems with any number of claimants if bilateral consistency or converse consistency is required in addition.*[2]

[2] This type of theorem can be formulated for any bankruptcy rule satisfying bilateral consistency and converse consistency.

Proof We know from Lemmas 3.2 and 3.3 that the proportional rule satisfies bilateral consistency and converse consistency. Let f be a bankruptcy rule on $\overline{\mathrm{BR}}^N$ satisfying the properties in the axiomatic characterization of the proportional rule on the class of two-claimant bankruptcy problems, and bilateral consistency or converse consistency. Then we have $f(E, c) = \mathrm{Prop}(E, c)$ for all $(E, c) \in \mathrm{BR}^S$ with $S \in 2^N$ and $|S| = 2$. Since the proportional rule satisfies bilateral consistency and converse consistency, we know from Lemma 3.1 that $f = \mathrm{Prop}$. $\square$

In particular, we can derive new characterizations of the proportional rule from the work of Dietzenbacher et al. (2016) using Theorem 3.4, by requiring the corresponding properties in the axiomatic characterizations for the class of two-claimant bankruptcy problems and adding bilateral or converse consistency.

Contrary to the class of TU-bankruptcy problems, the constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy multilateral consistency on the class of NTU-bankruptcy problems. This is shown by the following example.

Example 3 Let $N = \{1, 2, 3\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 + x_3^2 \leq 4\}$ and $c = (2, 2, 2)$. We have $u^E = (2, 2, 2)$ and

$$\mathrm{Prop}(E, c) = \mathrm{CREA}(E, c) = \mathrm{CREL}(E, c) = (1, 1, 1).$$

This means that

$$E_{\{1,2\}}^{\mathrm{Prop}(E,c)} = E_{\{1,2\}}^{\mathrm{CREA}(E,c)} = E_{\{1,2\}}^{\mathrm{CREL}(E,c)} = \left\{x \in \mathbb{R}^{\{1,2\}}_+ \mid x_1^2 + 2x_2 \leq 3\right\},$$

which implies that

$$\begin{aligned}
\mathrm{Prop}(E_{\{1,2\}}^{\mathrm{Prop}(E,c)}, c_{\{1,2\}}) &= (1, 1, \cdot)\,, \\
\mathrm{CREA}(E_{\{1,2\}}^{\mathrm{CREA}(E,c)}, c_{\{1,2\}}) &= \left(\tfrac{1}{2}\sqrt{15} - \tfrac{1}{2}\sqrt{3}, \tfrac{3}{4}\sqrt{5} - \tfrac{3}{4}, \cdot\right), \\
\text{and } \mathrm{CREL}(E_{\{1,2\}}^{\mathrm{CREL}(E,c)}, c_{\{1,2\}}) &= \left(\tfrac{1}{6}\sqrt{72\sqrt{3}-9} - \tfrac{1}{2}\sqrt{3}, \tfrac{1}{4}\sqrt{24\sqrt{3}-3} - \sqrt{3} + \tfrac{5}{4}, \cdot\right).
\end{aligned}$$

Hence, the constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy multilateral consistency.

However, we have

$$\begin{aligned}
E_{\{1,3\}}^{\mathrm{Prop}(E,c)} = E_{\{1,3\}}^{\mathrm{CREA}(E,c)} &= E_{\{1,3\}}^{\mathrm{CREL}(E,c)} \\
&= \left\{x \in \mathbb{R}^{\{1,3\}}_+ \mid x_1^2 + x_3^2 \leq 2\right\},
\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{Prop}(E_{\{1,3\}}^{\mathrm{Prop}(E,c)}, c_{\{1,3\}}) &= \mathrm{CREA}(E_{\{1,3\}}^{\mathrm{CREA}(E,c)}, c_{\{1,3\}})\\ &= \mathrm{CREL}(E_{\{1,3\}}^{\mathrm{CREL}(E,c)}, c_{\{1,3\}})\\ &= (1, \cdot, 1)\,.\end{aligned}$$

△

In Example 3, the constrained relative equal awards rule and the constrained relative equal losses rule do satisfy consistency on the domain of reduced problems for which the ratio of utopia values is equal to the ratio of utopia values in the original problem. This holds in general. We introduce the restricted consistency property to describe this type of bankruptcy rules.[3]

Definition 3.4 (*Restricted consistency*) A bankruptcy rule f on $\overline{\mathrm{BR}}^N$ satisfies *restricted consistency* if $f_S(E, c) = f(E_S^{f(E,c)}, c_S)$ for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N \backslash \{\emptyset\}$ for which $u^{E_S^{f(E,c)}} = t u_S^E$ for some $t \in [0, 1]$.

Note that both multilateral consistency and restricted consistency generalize the consistency notion for TU-bankruptcy rules.

Proposition 3.5 *The constrained relative equal awards rule satisfies restricted consistency.*

Proof Let $(E, c) \in \mathrm{BR}^N$ and let $S \in 2^N \backslash \{\emptyset\}$ be such that $u^{E_S^{\mathrm{CREA}(E,c)}} = t u_S^E$ for some $t \in [0, 1]$. We have $\mathrm{CREA}_i(E, c) = \min\{c_i, \alpha^{E,c} u_i^E\}$ for all $i \in S$ and

$$\begin{aligned}\mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S) &= (\min\{c_i, \alpha^{E_S^{\mathrm{CREA}(E,c)}, c_S} u_i^{E_S^{\mathrm{CREA}(E,c)}}\})_{i \in S}\\ &= (\min\{c_i, t\alpha^{E_S^{\mathrm{CREA}(E,c)}, c_S} u_i^E\})_{i \in S},\end{aligned}$$

where $\alpha^{E,c} \in [0, 1]$ is such that $\mathrm{CREA}(E, c) \in \mathrm{WP}(E)$ and $\alpha^{E_S^{\mathrm{CREA}(E,c)}, c_S} \in [0, 1]$ is such that

$$\mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{CREA}(E,c)}).$$

Since $\mathrm{CREA}_S(E, c) \in E_S^{\mathrm{CREA}(E,c)}$, we have $\mathrm{CREA}_S(E, c) \leq \mathrm{CREA}_S(E_S^{\mathrm{CREA}(E,c)}, c_S)$. Since E is nonleveled and

$$\left(\mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S), \mathrm{CREA}_{N \backslash S}(E, c)\right) \in E,$$

this means that $\mathrm{CREA}_S(E, c) = \mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S)$. Hence, the constrained relative equal awards rule satisfies restricted consistency. □

Proposition 3.6 *The constrained relative equal losses rule satisfies restricted consistency.*

[3] Peters et al. (1994) introduced a similar property for bargaining solutions.

Proof Let $(E, c) \in \mathrm{BR}^N$ with $E \neq \{0_N\}$ and let $S \in 2^N \setminus \{\emptyset\}$ be such that $u^{E_S^{\mathrm{CREL}(E,c)}} = tu_S^E$ for some $t \in [0, 1]$. We have $\mathrm{CREL}_i(E, c) = \max\{0, c_i - \beta^{E,c} u_i^E\}$ for all $i \in S$ and

$$\begin{aligned}\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S) &= (\max\{0, c_i - \beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} u_i^{E_S^{\mathrm{CREL}(E,c)}}\})_{i \in S} \\ &= (\max\{0, c_i - t\beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} u_i^E\})_{i \in S},\end{aligned}$$

where $\beta^{E,c} \in \mathbb{R}_+$ is such that $\mathrm{CREL}(E, c) \in \mathrm{WP}(E)$ and $\beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} \in \mathbb{R}_+$ is such that

$$\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{CREL}(E,c)}).$$

Since $\mathrm{CREL}_S(E, c) \in E_S^{\mathrm{CREL}(E,c)}$, we have $\mathrm{CREL}_S(E, c) \leq \mathrm{CREL}_S(E_S^{\mathrm{CREL}(E,c)}, c_S)$. Since E is nonleveled and

$$\left(\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S), \mathrm{CREL}_{N \setminus S}(E, c)\right) \in E,$$

this means that $\mathrm{CREL}_S(E, c) = \mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S)$. Hence, the constrained relative equal losses rule satisfies restricted consistency. $\square$

Converting reduced problems induced by leaving claimants into new bankruptcy problems for the remaining claimants tends to lose characteristics of the original problems. In particular, significant information on the interrelations of the remaining claimants is lost by the projection operation. Instead, the reduced problem could also be interpreted as the original bankruptcy problem where the payoffs of the leaving claimants have already been determined. In a sense, the leaving claimants leave a footprint behind on the original bankruptcy problem. To formalize this approach, we redefine bankruptcy rules assigning to any footprint bankruptcy problem an allocation for which the payoffs of the remaining claimants are bounded by their claims, and the leaving claimants are assigned their footprints.

A *footprint bankruptcy problem* is a quintuple (N, E, c, x, S) where $(E, c) \in \mathrm{BR}^N$ is a bankruptcy problem, $x \in \mathbb{R}_+^N$ is a vector of footprints, and $S \in 2^N \setminus \{\emptyset\}$ is a set of remaining claimants for which $(E_S^x, c_S) \in \mathrm{BR}^S$. Let FBR^N denote the class of all footprint bankruptcy problems with claimant set N. For convenience, a footprint bankruptcy problem is denoted by $(E, c, x, S) \in \mathrm{FBR}^N$ and $(E, c, x, N) \in \mathrm{FBR}^N$ is abbreviated to $(E, c) \in \mathrm{FBR}^N$.

A bankruptcy rule f on FBR^N assigns to any footprint bankruptcy problem $(E, c, x, S) \in \mathrm{FBR}^N$ a payoff allocation $f(E, c, x, S) \in \mathrm{WP}(E)$ for which

$$f_S(E, c, x, S) \leq c_S \text{ and } f_{N \setminus S}(E, c, x, S) = x_{N \setminus S}.$$

Note that $(E, c, f(E, c), S) \in \mathrm{FBR}^N$ for all $(E, c) \in \mathrm{BR}^N$, any $S \in 2^N \setminus \{\emptyset\}$, and any bankruptcy rule f on FBR^N.

The proportional rule Prop on FBR^N assigns to any $(E, c, x, S) \in \mathrm{FBR}^N$ the payoff allocation for which

$$\mathrm{Prop}_S(E, c, x, S) = \lambda^{E,c,x,S} c_S,$$

where $\lambda^{E,c,x,S} \in [0, 1]$ is such that $\mathrm{Prop}(E, c, x, S) \in \mathrm{WP}(E)$.

The constrained relative equal awards rule CREA on FBR^N assigns to any $(E, c, x, S) \in \mathrm{FBR}^N$ the payoff allocation for which

$$\mathrm{CREA}_S(E, c, x, S) = \left(\min\{c_i, \alpha^{E,c,x,S} u_i^E\}\right)_{i \in S},$$

where $\alpha^{E,c,x,S} \in [0, 1]$ is such that $\mathrm{CREA}(E, c, x, S) \in \mathrm{WP}(E)$.

The constrained relative equal losses rule CREL on FBR^N assigns to any $(E, c, x, S) \in \mathrm{FBR}^N$ with $E \neq \{0_N\}$ the payoff allocation for which

$$\mathrm{CREL}_S(E, c, x, S) = \left(\max\{0, c_i - \beta^{E,c,x,S} u_i^E\}\right)_{i \in S},$$

where $\beta^{E,c,x,S} \in \mathbb{R}_+$ is such that $\mathrm{CREL}(E, c, x, S) \in \mathrm{WP}(E)$.

We now introduce the footprint consistency property to describe bankruptcy rules which prescribe the same payoff allocation for the original bankruptcy problem as for any footprint bankruptcy problem in which the footprints equal the allocated payoffs.

Definition 3.5 (Footprint consistency). A bankruptcy rule f on FBR^N satisfies *footprint consistency* if $f(E, c) = f(E, c, f(E, c), S)$ for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N \backslash \{\emptyset\}$.

Inspired by Young (1987), we introduce the class of parametric bankruptcy rules where the payoff allocated to a claimant only depends on individual characteristics within the bankruptcy problem and a common parameter. It turns out that all parametric bankruptcy rules satisfy footprint consistency.

Definition 3.6 (*Parametric bankruptcy rule*) A bankruptcy rule f on FBR^N is *parametric* if there exists a function $r^f : \mathbb{R}^3_+ \to \mathbb{R}_+$, monotonic in its third argument, for which $f_S(E, c, x, S) = (r^f(c_i, u_i^E, \theta^{E,c,x,S}))_{i \in S}$ for all $(E, c, x, S) \in \mathrm{FBR}^N$ and some parameter $\theta^{E,c,x,S} \in \mathbb{R}_+$.

Theorem 3.7 *All parametric bankruptcy rules satisfy footprint consistency.*

Proof Let f be a parametric bankruptcy rule on FBR^N, let $(E, c) \in \mathrm{BR}^N$ and let $S \in 2^N \backslash \{\emptyset\}$. Then, we have $f_{N \backslash S}(E, c) = f_{N \backslash S}(E, c, f(E, c), S)$. Moreover, we have $f_i(E, c) = r^f(c_i, u_i^E, \theta^{E,c})$ and $f_i(E, c, f(E, c), S) = r^f(c_i, u_i^E, \theta^{E,c,f(E,c),S})$ for all $i \in S$. Since r^f is monotonic in its third argument, this means that $f_S(E, c) \leq f_S(E, c, f(E, c), S)$ or $f_S(E, c) \geq f_S(E, c, f(E, c), S)$. Since E is nonleveled, this implies that $f(E, c) = f(E, c, f(E, c), S)$. Hence, f satisfies footprint consistency. $\square$

Specific examples of parametric bankruptcy rules are the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule.

Corollary 3.8 *The proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule satisfy footprint consistency.*

Example 4 Let $N = \{1, 2, 3\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 + x_3^2 \leq 4\}$ and $c = (2, 2, 2)$ as in Example 3. We have

$$\begin{aligned}\mathrm{Prop}_{\{1,2\}}(E, c, \mathrm{Prop}(E, c), \{1, 2\}) &= \mathrm{CREA}_{\{1,2\}}(E, c, \mathrm{CREA}(E, c), \{1, 2\}) \\ &= \mathrm{CREL}_{\{1,2\}}(E, c, \mathrm{CREL}(E, c), \{1, 2\}) \\ &= (1, 1, \cdot).\end{aligned} \qquad \triangle$$

Future research could further explore footprint consistency and the class of parametric bankruptcy rules.

4 The relative adjustment principle

The contested garment principle for TU-bankruptcy rules (cf. Aumann and Maschler 1985) describes a standard solution for bankruptcy problems with two claimants where they first concede the minimal rights to each other, and subsequently divide the remaining estate equally. To adequately generalize this two-claimant solution to the relative adjustment principle for NTU-bankruptcy rules, we first study minimal rights in NTU-bankruptcy problems.

The minimal right of a claimant in a TU-bankruptcy problem is defined as the remaining part of the estate when all other claimants are allocated their claims (cf. Curiel et al. 1987). Following Estévez-Fernández et al. (2019), we define the minimal right of a claimant in an NTU-bankruptcy problem as the maximal individual payoff within the estate when all other claimants are allocated their claims.

Let $(E, c) \in \mathrm{BR}^N$. The vector of *minimal rights* $m(E, c) \in \mathbb{R}^N_+$ is, for all $i \in N$, defined by

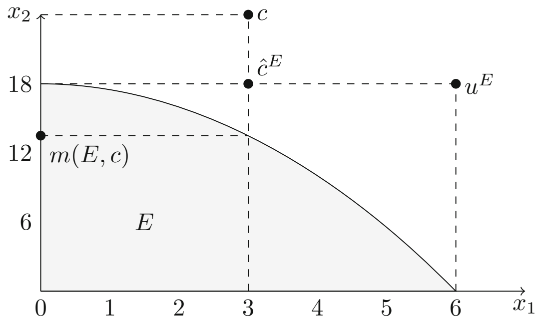
$$m_i(E, c) = \begin{cases} \max\{x \mid (x, c_{N\setminus\{i\}}) \in E\} & \text{if } (0, c_{N\setminus\{i\}}) \in \mathrm{E}; \\ 0 & \text{if } (0, c_{N\setminus\{i\}}) \notin \mathrm{E}. \end{cases}$$

We have $m(E, c) \in E$ and $m(E, c) \leq c$, which means that

$$((E - \{m(E, c)\})_+, c - m(E, c)) \in \mathrm{BR}^N.$$

Moreover, we have $m(E, c) \leq f(E, c) \leq \hat{c}^E$ for any bankruptcy rule f on BR^N.

Example 5 Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $m(E, c) = (0, 13\frac{1}{2})$. This is illustrated as follows.



△

The following lemma derives some elementary relations between truncated claims and minimal rights.

Lemma 4.1 *Let* $(E, c) \in \mathrm{BR}^N$. *Then*

(i) $\widehat{\hat{c}^E}^E = \hat{c}^E$;
(ii) $m((E - \{m(E, c)\})_+, c - m(E, c)) = 0_N$;
(iii) $m(E, \hat{c}^E) = m(E, c)$;
(iv) $\widehat{\text{c-m(E,c)}}^{(E-\{m(E,c)\})_+} = \hat{c}^E - m(E, c)$.

Proof (i) Let $i \in N$. Then

$$\begin{aligned}\widehat{\hat{c}^E_i}^E &= \min\{\hat{c}_i^E, u_i^E\} = \min\{\min\{c_i, u_i^E\}, u_i^E\}\\ &= \min\{c_i, u_i^E\} = \hat{c}_i^E.\end{aligned}$$

(ii) Let $i \in N$. Suppose that $m_i((E - \{m(E, c)\})_+, c - m(E, c)) > 0$. Then

$$\big(m_i((E - \{m(E, c)\})_+, c - m(E, c)), (c - m(E, c))_{N\setminus\{i\}}\big) \in (E - \{m(E, c)\})_+.$$

This means that

$$\big(m_i((E - \{m(E, c)\})_+, c - m(E, c)) + m_i(E, c), c_{N\setminus\{i\}}\big) \in E.$$

This contradicts the definition of $m_i(E, c)$.

(iii) Let $i \in N$. If $\hat{c}^E_{N\setminus\{i\}} = c_{N\setminus\{i\}}$, then $m_i(E, \hat{c}^E) = m_i(E, c)$. If $\hat{c}^E_{N\setminus\{i\}} \neq c_{N\setminus\{i\}}$, then $(0, c_{N\setminus\{i\}}) \notin E$, so $m_i(E, \hat{c}^E) = 0 = m_i(E, c)$.

(iv) Let $i \in N$. If $m_{N\setminus\{i\}}(E, c) = 0_{N\setminus\{i\}}$, then $u_i^{(E-\{m(E,c)\})_+} = u_i^E - m_i(E, c)$ and

$$\begin{aligned}\widehat{(c-m(E, c))}_i^{(E-\{m(E,c)\})_+} &= \min\left\{c_i - m_i(E, c), u_i^{(E-\{m(E,c)\})_+}\right\}\\ &= \min\{c_i - m_i(E, c), u_i^E - m_i(E, c)\}\\ &= \min\{c_i, u_i^E\} - m_i(E, c)\\ &= \hat{c}_i^E - m_i(E, c).\end{aligned}$$

Suppose that there exists a $j \in N\backslash\{i\}$ for which $m_j(E,c) > 0$. Then $\hat{c}_i^E = c_i$ and $(m_j(E,c), c_{N\backslash\{j\}}) \in E$. Since E is comprehensive and $m(E,c) \leq c$, this means that $(c_i, m_{N\backslash\{i\}}(E,c)) \in E$, so $(c_i - m_i(E,c), 0_{N\backslash\{i\}}) \in (E - \{m(E,c)\})_+$. This implies that $u_i^{(E-\{m(E,c)\})_+} \geq c_i - m_i(E,c)$ and

$$\begin{aligned}\widehat{(c - m(E,c))}_i^{(E-\{m(E,c)\})_+} &= \min\left\{c_i - m_i(E,c), u_i^{(E-\{m(E,c)\})_+}\right\} \\ &= c_i - m_i(E,c) \\ &= \hat{c}_i^E - m_i(E,c).\end{aligned}$$

□

The minimal rights first property requires that first allocating minimal rights and subsequently applying the bankruptcy rule to the remaining problem leads to the same payoff allocation as direct application of the bankruptcy rule to the original problem.

Definition 4.1 (*Minimal rights first*) A bankruptcy rule f on BR^N satisfies *minimal rights first* if

$$f(E,c) = m(E,c) + f((E - \{m(E,c)\})_+, c - m(E,c))$$

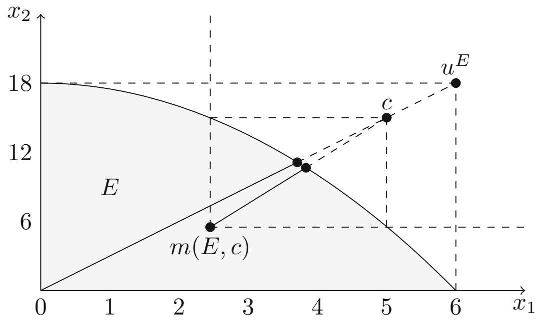
for all $(E,c) \in \mathrm{BR}^N$.

The following example shows that the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule do not satisfy minimal rights first.

Example 6 Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (5,15)$. We have $u^E = (6,18)$ and $m(E,c) = (\sqrt{6}, 5\frac{1}{2})$. Let f be a bankruptcy rule on BR^N satisfying relative symmetry, e.g. the proportional rule, the constrained relative equal awards rule, or the constrained relative equal losses rule. Then

$$f(E,c) \neq m(E,c) + f((E - \{m(E,c)\})_+, c - m(E,c)).$$

This is illustrated as follows.



△

Since the constrained relative equal awards rule and the constrained relative equal losses rule are dual bankruptcy rules (cf. Dietzenbacher et al. 2016), and the constrained relative equal awards rule satisfies truncation invariance, this means that minimal rights first and truncation invariance are not dual properties, in contrast to the TU-bankruptcy context (cf. Herrero and Villar 2001).

Inspired by Thomson and Yeh (2008), we introduce two operators which 'force' bankruptcy rules to satisfy truncation invariance and minimal rights first. Let $\mathcal{F}$ denote the space of all bankruptcy rules on BR^N. The *truncation operator* $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ assigns to any bankruptcy rule $f \in \mathcal{F}$ the bankruptcy rule $\mathcal{T}(f) \in \mathcal{F}$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathcal{T}(f)(E, c) = f(E, \hat{c}^E).$$

The *minimal rights operator* $\mathcal{M} : \mathcal{F} \to \mathcal{F}$ assigns to any bankruptcy rule $f \in \mathcal{F}$ the bankruptcy rule $\mathcal{M}(f) \in \mathcal{F}$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathcal{M}(f)(E, c) = m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)).$$

Note that both operators are well-defined. We have $f = \mathcal{T}(f)$ if and only if $f \in \mathcal{F}$ satisfies truncation invariance, and $f = \mathcal{M}(f)$ if and only if $f \in \mathcal{F}$ satisfies minimal rights first. In particular, this means that CREA $= \mathcal{T}$(CREA).

The next theorem studies some consequences of the truncation operator and the minimal rights operator for the properties of the bankruptcy rules to which they are applied.

Theorem 4.2 *Let $f \in \mathcal{F}$ be a bankruptcy rule.*

(i) *Then $\mathcal{T}(f)$ satisfies truncation invariance.*
(ii) *Then $\mathcal{M}(f)$ satisfies minimal rights first.*
(iii) *If f satisfies relative symmetry, then $\mathcal{T}(f)$ satisfies relative symmetry.*
(iv) *If f satisfies truncation invariance, then $\mathcal{M}(f)$ satisfies truncation invariance.*
(v) *If f satisfies minimal rights first, then $\mathcal{T}(f)$ satisfies minimal rights first.*

Proof (i) Let $(E, c) \in \mathrm{BR}^N$. Then

$$\mathcal{T}(f)(E, \hat{c}^E) = f(E, \widehat{\hat{c}^E}^E) = f(E, \hat{c}^E) = \mathcal{T}(f)(E, c),$$

where the second equality follows from Lemma 4.1(i).

(ii) Let $(E, c) \in \mathrm{BR}^N$. Then

$$\begin{aligned} &m(E, c) + \mathcal{M}(f)((E - \{m(E, c)\})_+, c - m(E, c)) \\ &\quad = m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)) \\ &\quad = \mathcal{M}(f)(E, c), \end{aligned}$$

where the first equality follows from Lemma 4.1(ii).

(iii) Assume that f satisfies relative symmetry. Let $(E, c) \in \mathrm{BR}^N$ and let $i, j \in N$ be such that $c_i u_j^E = c_j u_i^E$. Then

$$\begin{aligned}\hat{c}_i^E u_j^E &= \min\{c_i, u_i^E\} u_j^E = \min\{c_i u_j^E, u_i^E u_j^E\} = \min\{c_j u_i^E, u_j^E u_i^E\} \\ &= \min\{c_j, u_j^E\} u_i^E = \hat{c}_j^E u_i^E.\end{aligned}$$

Since f satisfies relative symmetry, this means that

$$\mathcal{T}(f)_i(E, c) u_j^E = f_i(E, \hat{c}^E) u_j^E = f_j(E, \hat{c}^E) u_i^E = \mathcal{T}(f)_j(E, c) u_i^E.$$

(iv) Assume that f satisfies truncation invariance. Let $(E, c) \in \mathrm{BR}^N$. Then

$$\begin{aligned}\mathcal{M}(f)(E, \hat{c}^E) &= m(E, \hat{c}^E) + f((E - \{m(E, \hat{c}^E)\})_+, \hat{c}^E - m(E, \hat{c}^E)) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, \widehat{(c - m(E, c))}^{(E - \{m(E,c)\})_+}) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)) \\ &= \mathcal{M}(f)(E, c),\end{aligned}$$

where the second equality follows from Lemma 4.1(iii), the third equality follows from Lemma 4.1(iv), and the fourth equality follows from f satisfying truncation invariance.

(v) Assume that f satisfies minimal rights first. Let $(E, c) \in \mathrm{BR}^N$. Then

$$\begin{aligned}& m(E, c) + \mathcal{T}(f)((E - \{m(E, c)\})_+, c - m(E, c)) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, \widehat{(c - m(E, c))}^{(E - \{m(E,c)\})_+}) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) \\ &= m(E, \hat{c}^E) + f((E - \{m(E, \hat{c}^E)\})_+, \hat{c}^E - m(E, \hat{c}^E)) \\ &= f(E, \hat{c}^E) \\ &= \mathcal{T}(f)(E, c),\end{aligned}$$

where the second equality follows from Lemma 4.1(iv), the third equality follows from Lemma 4.1(iii), and the fourth equality follows from f satisfying minimal rights first. □

The purpose of Theorem 4.2 is twofold. First, it shows that the truncation operator and the minimal rights operator indeed ‘force’ bankruptcy rules to satisfy truncation invariance and minimal rights first, respectively. Second, it studies the preservation of properties under the truncation operator and the minimal rights operator. Both operators preserve truncation invariance and minimal rights first. Relative symmetry is preserved under the truncation operator, but Example 6 shows that it is not preserved under the minimal rights operator.

Let $f \in \mathcal{F}$. From Theorem 4.2 we know that $\mathcal{T}(f)$ satisfies truncation invariance and $\mathcal{M}(f)$ satisfies minimal rights first, which means that

$$\mathcal{T}(\mathcal{T}(f)) = \mathcal{T}(f) \text{ and } \mathcal{M}(\mathcal{M}(f)) = \mathcal{M}(f).$$

By the preservation of properties, $\mathcal{T}(\mathcal{M}(f))$ and $\mathcal{M}(\mathcal{T}(f))$ both satisfy truncation invariance and minimal rights first, which means that

$$\begin{aligned}&\mathcal{T}(\mathcal{M}(\mathcal{T}(f))) = \mathcal{T}(\mathcal{M}(\mathcal{M}(f))) = \mathcal{T}(\mathcal{M}(f))\\ &\text{and}\mathcal{M}(\mathcal{T}(\mathcal{T}(f))) = \mathcal{M}(\mathcal{T}(\mathcal{M}(f))) = \mathcal{M}(\mathcal{T}(f)).\end{aligned}$$

Hence, nothing changes when one of the operators is applied more than once. However, the two operators can be combined to obtain a bankruptcy rule which satisfies both truncation invariance and minimal rights first. The following proposition shows that the order in which the operators are applied does not matter.

Proposition 4.3 *Let $f \in \mathcal{F}$. Then $\mathcal{T}(\mathcal{M}(f)) = \mathcal{M}(\mathcal{T}(f))$.*

Proof Let $(E, c) \in \mathrm{BR}^N$. We can write

$$\begin{aligned}\mathcal{T}(\mathcal{M}(f))(E, c) &= \mathcal{M}(f)(E, \hat{c}^E)\\ &= m(E, \hat{c}^E) + f((E - \{m(E, \hat{c}^E)\})_+, \hat{c}^E - m(E, \hat{c}^E))\\ &= m(E, c) + f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c))\\ &= m(E, c) + f((E - \{m(E, c)\})_+, \widehat{(c-m(E, c))}^{(E-\{m(E,c)\})_+})\\ &= m(E, c) + \mathcal{T}(f)((E - \{m(E, c)\})_+, c - m(E, c))\\ &= \mathcal{M}(\mathcal{T}(f))(E, c),\end{aligned}$$

where the third equality follows from Lemma 4.1(iii) and the fourth equality follows from Lemma 4.1(iv). □

The bankruptcy rule $\mathcal{T}(\mathcal{M}(f))$ is called the *adjusted counterpart* of the rule $f \in \mathcal{F}$. Three examples of adjusted bankruptcy rules are given by the adjusted proportional rule $\mathcal{T}(\mathcal{M}(\mathrm{Prop}))$,[4] the adjusted constrained relative equal awards rule $\mathcal{T}(\mathcal{M}(\mathrm{CREA}))$, and the adjusted constrained relative equal losses rule $\mathcal{T}(\mathcal{M}(\mathrm{CREL}))$. On the class of bankruptcy problems with two claimants, these three adjusted bankruptcy rules coincide. This standard solution for two-claimant bankruptcy problems is called the relative adjustment principle.[5]

Definition 4.2 (*Relative adjustment principle*) The *relative adjustment principle* RAP on BR^N with $|N| = 2$ assigns to any $(E, c) \in \mathrm{BR}^N$ with $|N| = 2$ the payoff allocation

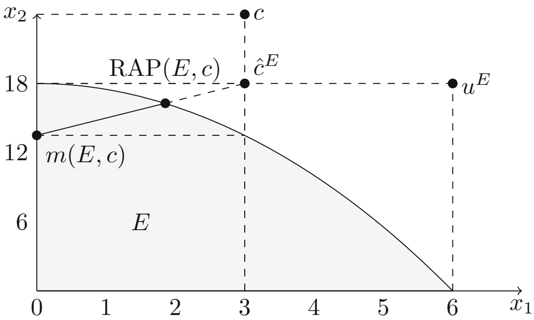
$$\mathrm{RAP}(E, c) = m(E, c) + \rho^{E,c}\left(\hat{c}^E - m(E, c)\right),$$

[4] The adjusted proportional rule for TU-bankruptcy problems was introduced by Curiel et al. (1987). In the context of bargaining problems with claims (cf. Chun and Thomson 1992), a similar adjusted proportional rule was introduced by Herrero (1997).

[5] For TU-bankruptcy problems, Aumann and Maschler (1985) called this standard solution the contested garment principle. Later, Thomson (2003) named it the concede-and-divide principle.

where $\rho^{E,c} \in [0, 1]$ is such that $\mathrm{RAP}(E, c) \in \mathrm{WP}(E)$.[6]

Example 7 Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1 and Example 5. We have $\hat{c}^E = (3, 18)$ and $m(E, c) = (0, 13\frac{1}{2})$. This means that $\mathrm{RAP}(E, c) = (\frac{3}{2}\sqrt{5} - 1\frac{1}{2}, \frac{9}{4}\sqrt{5} + 11\frac{1}{4})$. This is illustrated as follows.



△

In order to axiomatically study the relative adjustment principle, we introduce the class of simple bankruptcy problems.

Definition 4.3 (*Simple bankruptcy problem*) A bankruptcy problem $(E, c) \in \mathrm{BR}^N$ is called *simple* if $\hat{c}^E = c$ and $m(E, c) = 0_N$.

Let SBR^N denote the class of all simple bankruptcy problems with claimant set N.

Lemma 4.4 *Let* $(E, c) \in \mathrm{BR}^N$. *Then* $((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) \in \mathrm{SBR}^N$.

Proof We can write

$$\begin{aligned}
\widehat{(\hat{c}^E - m(E,c))}^{(E-\{m(E,c)\})_+} &= \widehat{(\hat{c}^E - m(E,\hat{c}^E))}^{(E-\{m(E,\hat{c}^E)\})_+} \\
&= \widehat{\hat{c}^E}^E - m(E,\hat{c}^E) \\
&= \hat{c}^E - m(E,c),
\end{aligned}$$

where the first equality follows from Lemma 4.1(iii), the second equality follows from Lemma 4.1(iv), and the third equality follows from Lemma 4.1(i) and Lemma 4.1(iii). Moreover, we can write

$$m((E-\{m(E,c)\})_+, \hat{c}^E - m(E,c)) = m((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) = 0_N,$$

where the first equality follows from Lemma 4.1(iii) and the second equality follows from Lemma 4.1(ii). □

[6] For an arbitrary number of claimants, this formula corresponds to the adjusted proportional rule.

A bankruptcy rule satisfies the *simple counterpart* of a property if it satisfies that property on the class of simple bankruptcy problems. For instance, a bankruptcy rule $f \in \mathcal{F}$ satisfies *simple relative symmetry* if $f_i(E,c)u_j^E = f_j(E,c)u_i^E$ for all $(E,c) \in \mathrm{SBR}^N$ and any $i, j \in N$ with $c_i u_j^E = c_j u_i^E$. Note that all bankruptcy rules satisfy *simple truncation invariance* and *simple minimal rights first*.

If a bankruptcy rule satisfies a property, then Lemma 4.4 implies that its adjusted counterpart satisfies the simple counterpart of that property. For instance, the adjusted counterpart of any relatively symmetric bankruptcy rule satisfies simple relative symmetry. Inspired by Dagan (1996), we axiomatically characterize the relative adjustment principle by simple relative symmetry, truncation invariance, and minimal rights first. In particular, this means that the adjusted counterpart of any relatively symmetric bankruptcy rule coincides with the relative adjustment principle on the class of bankruptcy problems with two claimants.

Theorem 4.5 *The relative adjustment principle is the unique two-claimant bankruptcy rule satisfying simple relative symmetry, truncation invariance, and minimal rights first.*

Proof First, we show that the relative adjustment principle satisfies simple relative symmetry, truncation invariance, and minimal rights first. Let $(E,c) \in \mathrm{SBR}^N$ with $|N| = 2$ and let $i, j \in N$ be such that $c_i u_j^E = c_j u_i^E$. Since $\hat{c}^E = c$ and $m(E,c) = 0_N$, we can write

$$\begin{aligned}
\mathrm{RAP}_i(E,c)u_j^E &= \left(m_i(E,c) + \rho^{E,c}\left(\hat{c}_i^E - m_i(E,c)\right)\right)u_j^E \\
&= \rho^{E,c} c_i u_j^E \\
&= \rho^{E,c} c_j u_i^E \\
&= \left(m_j(E,c) + \rho^{E,c}\left(\hat{c}_j^E - m_j(E,c)\right)\right)u_i^E \\
&= \mathrm{RAP}_j(E,c)u_i^E.
\end{aligned}$$

Hence, RAP satisfies simple relative symmetry. Let $(E,c) \in \mathrm{BR}^N$ with $|N| = 2$. We can write

$$\begin{aligned}
\mathrm{RAP}(E,\hat{c}^E) &= m(E,\hat{c}^E) + \rho^{E,\hat{c}^E}\left(\widehat{\hat{c}^E}^E - m(E,\hat{c}^E)\right) \\
&= m(E,c) + \rho^{E,\hat{c}^E}\left(\hat{c}^E - m(E,c)\right),
\end{aligned}$$

where the second equality follows from Lemmas 4.1(i) and 4.1(iii). Since E is non-leveled, this means that $\mathrm{RAP}(E,c) = \mathrm{RAP}(E,\hat{c}^E)$. Hence, RAP satisfies truncation invariance.

Let $(E, c) \in \mathrm{BR}^N$ with $|N| = 2$. We can write

$$\begin{aligned} & m(E,c) + \mathrm{RAP}((E - \{m(E,c)\})_+, c - m(E,c)) \\ & \quad = m(E,c) + \rho^{(E-\{m(E,c)\})_+, c-m(E,c)} \left(\widehat{c\text{-}m(E,c)}^{(E-\{m(E,c)\})_+} \right) \\ & \quad = m(E,c) + \rho^{(E-\{m(E,c)\})_+, c-m(E,c)} \left(\hat{c}^E - m(E,c) \right), \end{aligned}$$

where the first equality follows from Lemma 4.1(ii) and the second equality follows from Lemma 4.1(iv). Since E is nonleveled, this means that

$$\mathrm{RAP}(E,c) = m(E,c) + \mathrm{RAP}((E - \{m(E,c)\})_+, c - m(E,c)).$$

Hence, RAP satisfies minimal rights first.

Second, we show that there is a unique two-claimant bankruptcy rule satisfying simple relative symmetry, truncation invariance, and minimal rights first. Let f be a bankruptcy rule on BR^N with $|N| = 2$ satisfying simple relative symmetry, truncation invariance, and minimal rights first. Let $(E, c) \in \mathrm{BR}^N$ with $|N| = 2$. Since f satisfies truncation invariance and minimal rights first, we have

$$f(E,c) = m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)).$$

We know from Lemma 4.4 that $((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \in \mathrm{SBR}^N$. Let $i \in N$ and let $j \in N \backslash \{i\}$. We can write

$$\begin{aligned} u_i^{(E-\{m(E,c)\})_+} &= \max\{x_i \mid x \in (E - \{m(E,c)\})_+\} \\ &= \max\{x_i \mid (x_i + m_i(E,c), m_j(E,c)) \in E\} \\ &= \begin{cases} u_i^E - m_i(E,c) & \text{if } \ m_j(E,c) = 0; \\ c_i - m_i(E,c) & \text{if } \ m_j(E,c) > 0 \end{cases} \\ &= \begin{cases} u_i^E - m_i(E,c) & \text{if } \ \hat{c}_i^E = u_i^E; \\ c_i - m_i(E,c) & \text{if } \ \hat{c}_i^E = c_i \end{cases} \\ &= \hat{c}_i^E - m_i(E,c). \end{aligned}$$

This means that

$$\left(\hat{c}_i^E - m_i(E,c) \right) u_j^{(E-\{m(E,c)\})_+} = \left(\hat{c}_j^E - m_j(E,c) \right) u_i^{(E-\{m(E,c)\})_+}.$$

Since f satisfies simple relative symmetry, this implies that

$$f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) = t \left(\hat{c}^E - m(E,c) \right)$$

for some $t \in [0, 1]$. We can write

$$\begin{aligned} f(E,c) &= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\ &= m(E,c) + t\left(\hat{c}^E - m(E,c)\right). \end{aligned}$$

Since E is nonleveled, this means that

$$f(E,c) = m(E,c) + \rho^{E,c}\left(\hat{c}^E - m(E,c)\right).$$

Hence, $f = \text{RAP}$. □

Future research could study other characterizations of the relative adjustment principle inspired by results for the contested garment principle based on self-duality (cf. Dagan 1996), securement (cf. Moreno-Ternero and Villar 2004), and lower/upper securement (cf. Moreno-Ternero and Villar 2006).

Corollary 4.6 *The adjusted proportional rule, the adjusted constrained relative equal awards rule, and the adjusted constrained relative equal losses rule coincide with the relative adjustment principle on the class of bankruptcy problems with two claimants.*

Future research could study generalizations of other bankruptcy rules which coincide with the relative adjustment principle on the class of two-claimant TU-bankruptcy problems, such as the random arrival rule (cf. O'Neill 1982), the minimal overlap rule (cf. O'Neill (1982)), and the Talmud rule (cf. Aumann and Maschler 1985).

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